

# How to Add Divergent Series

## Abstract

In string theory, we have to calculate  $1 + 2 + 3 + \dots = -\frac{1}{12}$ .

“Divergent series are on the whole devil’s work, and it is a shame that one dares to found any proof on them. One can get out of them what one wants if one uses them, and it is they which have made so much unhappiness and so many paradoxes. Can one think of anything more appalling than to say that

$$0 = 1 - 2^n + 3^n - 4^n + \dots$$

where  $n$  is a positive number. Here’s something to laugh at, friends.” - Niels Henrik Abel (1826)

## 1 $1 + 2 + 3 + 4 + \dots$

A Use the derivative of geometric series

$$\begin{aligned} 1 + 2x + 3x^2 + \dots &= \frac{d}{dx} (1 + x + x^2 + \dots) = \frac{d}{dx} \frac{1}{1-x} = \frac{-1}{(1-x)^2} \\ &= \frac{-1}{(\epsilon + \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6} + \dots)^2} = \frac{1}{\epsilon^2} \cdot \frac{-1}{(1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{6} + \dots)^2} \\ &= \frac{-1}{\epsilon^2} (1 - \epsilon + \frac{\epsilon^2}{12} + \dots) \end{aligned}$$

And let  $x = e^\epsilon$ . Or maybe try

$$x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2}$$

to get the same answer. The finite term is  $-\frac{1}{12}$ .

## 2 $1 - 2 + 3 - 4 + \dots$

### 2.1 $1 - 1 + 1 - 1 + \dots$

The partial sums alternate  $1, 0, 1, 0, 1, 0, \dots$  we can split the difference and say it’s  $\frac{1}{2}$ .

$$\begin{aligned} x &= 1 - 1 + 1 - 1 + \dots \\ &= 1 - x \end{aligned}$$

So that  $2x = 1$  or  $x = \frac{1}{2}$ . Switching signs of infinitely many terms is only kinda-sorta legal.

### 2.2 $1 - 2 + 3 - 4 + \dots$

A Ernesto Cesaro points out [2] that  $(1 - 1 + 1 - 1 + \dots)^2 = 1 - 2 + 3 - 4 + \dots = \frac{1}{4}$ .

**B** Or we could say the partial sums are  $1, -1, 2, -2, 3, -3, \dots$  and the averages of these partial sums are  $1, 0, \frac{2}{3}, 0, \frac{3}{5}, 0, \frac{4}{7}, \dots$ . This is still not convergent. What about taking averaging again? It should be the average between 0 and  $\frac{1}{2}$  giving us  $\frac{1}{4}$ .

**C** We could take the Euler transform of  $a_0 - a_1 + a_2 - a_3 + \dots$

$$\frac{a_0}{2} - \frac{\Delta a_0}{4} + \frac{\Delta^2 a_0}{8} - \dots = \frac{1}{2}a_0 - \frac{1}{4}(a_0 - a_1) + \frac{1}{8}(a_0 - 2a_1 + a_2) + \frac{1}{16}(a_0 - 3a_1 + 3a_2 - a_3) - \dots$$

What's implicit are generalizations of the geometric series:

$$\frac{\binom{k}{2}}{2^k} + \frac{\binom{k+1}{2}}{2^{k+1}} + \frac{\binom{k+2}{2}}{2^{k+2}} + \dots = 2$$

### 3 $1 + 2 + 4 + 8 + \dots$

It has been known for a while how to add the divergent geometric series

$$x = 1 + 2 + 4 + 8 + \dots$$

If we chop off the first term

$$x = 1 + 2(1 + 2 + 4 + \dots) = 1 + 2x$$

Therefore  $x = -1$ . Or we can just use the geometric series formula

$$x = \frac{1}{1-2} = -1$$

In 1890's Hensel came up with the  $p$ -adic numbers (here  $p = 2$ ). Introduce a norm

$$|n| = |2^k m|_2 = \frac{1}{2^k}$$

This is indeed a norm and defines a metric on the integers,  $\mathbb{Z}$ . The completion  $\mathbb{Z}_2$  are the 2-adic numbers.

## References

- [1] Joe Polchinski. "Little Book of String" online.
- [2] Wikipedia "1 - 2 + 3 - 4 + ...", "1 + 2 + 3 + 4 + ...".